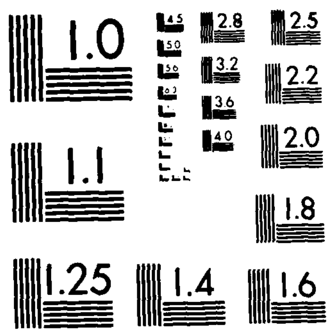
AD-A192 039 COMPUTATION OF FILTERS BY SAMPLING AND QUANTIZATION(U) 1/1
NORTH CAROLINA UNIV AT CHAPEL HILL CENTER FOR
STOCHASTIC PROCESSES A H KOREZLIOGLU SEP 87 TR-208
UNCLASSIFIED AFOSR-TR-88-0356 F49620-85-C-0144 F/G 12/3 NL



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS 1963 A

AD-A192 839

REPORT DOCUMENTATION PAGE

DTIC FILE COPY

1a. REPORT SECURITY CLASSIFICATION Unclassified	1b. RESTRICTIVE MARKINGS
2a. SECURITY CLASSIFICATION AUTHORITY	3. DISTRIBUTION/AVAILABILITY OF REPORT Approved for public release; distribution unlimited.
2b. DECLASSIFICATION / DOWNGRADING SCHEDULE	
4. PERFORMING ORGANIZATION REPORT NUMBER(S) Technical Report No. 208	5. MONITORING ORGANIZATION REPORT NUMBER(S) AFOSR-TR- 88-0356
6a. NAME OF PERFORMING ORGANIZATION University of North Carolina — 6b. OFFICE SYMBOL (If applicable)	7a. NAME OF MONITORING ORGANIZATION AFOSR/NM
6c. ADDRESS (City, State, and ZIP Code) Statistics Dept. 321-A Phillips Hall 039-A Chapel Hill, NC 27514	7b. ADDRESS (City, State, and ZIP Code) AFOSR/NM Bldg 410 Bolling AFB DC 20332-6448
8a. NAME OF FUNDING/SPONSORING ORGANIZATION AFOSR — 8b. OFFICE SYMBOL (If applicable) NM	9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER ~~AFOSR No~~. F49620 85C 0144.
8c. ADDRESS (City, State, and ZIP Code) AFOSR/NM Bldg 410 Bolling AFB DC 20332-6448	10. SOURCE OF FUNDING NUMBERS

PROGRAM ELEMENT NO.	PROJECT NO.	TASK NO.	WORK UNIT ACCESSION NO
61102F	2304		

11. TITLE (Include Security Classification)
Computation of filters by sampling and quantization

12. PERSONAL AUTHOR(S)
Korezlioglu, A.H.

13a. TYPE OF REPORT	13b. TIME COVERED	14. DATE OF REPORT (Year, Month, Day)	15. PAGE COUNT
preprint	FROM 9/87 TO 8/88	September 1987	29

16. SUPPLEMENTARY NOTATION
N/A

17. COSATI CODES			18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)
FIELD	GROUP	SUB-GROUP	Key Words & Phrases: N/A

19. ABSTRACT (Continue on reverse if necessary and identify by block number)

Various approximation procedures of filters by sampling and quantization are considered for effective computation. The corresponding approximation degrees are estimated without the boundedness condition on the modulated signal.

DTIC
ELECTE
APR 0 4 1988
S D
D

20. DISTRIBUTION/AVAILABILITY OF ABSTRACT ☒ UNCLASSIFIED/UNLIMITED ☐ SAME AS RPT. ☐ DTIC USERS	21. ABSTRACT SECURITY CLASSIFICATION Unclassified/unlimited
22a. NAME OF RESPONSIBLE INDIVIDUAL Maj. Brian Woodruff	22b. TELEPHONE (Include Area Code) (202) 767-5026 — 22c. OFFICE SYMBOL NM

DD FORM 1473, 84 MAR — 83 APR edition may be used until exhausted. All other editions are obsolete.

SECURITY CLASSIFICATION OF THIS PAGE
Unclassified/Unlimited

AFOSR-TR- 88-0356

CENTER FOR STOCHASTIC PROCESSES

Department of Statistics
University of North Carolina
Chapel Hill, North Carolina

COMPUTATION OF FILTERS BY SAMPLING AND QUANTIZATION

by

A.H. Korezlioglu

Technical Report No. 208

September 1987

88 3 31 028

COMPUTATION OF FILTERS BY

SAMPLING AND QUANTIZATION

A.H. Korezlioglu

Center for Stochastic Processes
Department of Statistics
University of North Carolina
Chapel Hill, NC 27599-3260

Abstract: Various approximation procedures of filters by sampling and quantization are considered for effective computation. The corresponding approximation degrees are estimated without the boundedness condition on the modulated signal.

Permanent address: Ecole Nationale Supérieure des Télécommunications
46, rue Barrault, 75634 Paris, Cedex 13, France

Research supported by Air Force Office of Scientific Research Contract No. F49620 85C 0144.

1. INTRODUCTION

The stochastic filtering theory has been a great source of motivation for research in the field of stochastic processes and there is abundant literature on the subject, (cf. e.g. [7], [8], [10], [12], [16], [23], [24] and references contained therein). One of the main motivation of research has been the solution of the filtering equation. Since in the linear case the filtering equation has an explicit solution, the early attempts of approximate computations in the nonlinear case were based on the linearization techniques which gave rise to what is called the extended Kalman-Bucy filter; (cf. [8]). The corresponding approximations only aimed at the computation of the conditional expectation of the state variable at a certain instant in terms of the observations up to a certain time. The computations for the extended Kalman-Bucy filters are easy (particularly in the discrete-time case), but one inconvenience of the method lies in that the accuracy of the approximation can not be evaluated. The reference probability method ([1], [22], [25]) allowing the derivation of the Zakai equation, has opened a new way to considering the filtering problem. Although there is no method for actually solving the Zakai equation much has been added to the qualitative study of the filtering process. On the other hand, the reference probability method has been very useful in developing approximation procedures ([3], [11], [13], [19], [20]) some of which are exposed here.

After a short introduction to stochastic filtering, we present the approximation procedures based on the periodic sampling of both the observation and the signal. Approximate filters obtained there correspond to the discrete time filtering of Markov chains with values in some $\mathbb{R}^q$-space and are expressed in terms of integrals depending on a continuous parameter, which render the numerical computation difficult. In order to eliminate this difficulty and to

help computations we propose a quantization of the signal and the observation processes. The proposed approximation schemes follow those of [11] and go as follows in the decreasing order of difficulty for computations. i) Periodic sampling of the observation, ii) periodic sampling of the signal, iii) Euler approximation of the signal, iv) quantization of the signal,
v) quantization of the observation. For each step the expression of the approximate filter is given and the corresponding approximation degree is estimated.

The estimation of the approximation degrees was considered in [3], [11], [19] and [20], but always with the boundedness condition on the modulated signal h or some restrictive conditions on the likelihood ratio excluding the linear model from the approximation procedures. The method proposed there does not necessitate such conditions; it only requires, as in the aforementioned references, Lipschitz and linear growth conditions that are necessary for benefiting from the continuity of the signal process and of its various approximations.

In order to make our approach as general as possible we choose a model in which the modulated signal and the drift and diffusion coefficients are time dependent.

The approximation degrees obtained here are of the order of $\sqrt{\delta}$, as in [3] and [11], where δ is the sampling period but, because of the widening of the conditions, the bounds are better. In [19] and [20], approximation degrees of the order of δ were obtained under more regularity conditions on the system and in a slightly different frame.

Section 2 is a short presentation of the reference probability method without derivation of the Zakai equation, since we only approximate the Kallianpur-Striebel formula.

Section 3 presents the first three approximation schemes and the corresponding recursive equations for the filters.

Section 4 presents the method of estimation of approximation degrees and gives the bounds for the first and second approximations.

Section 5 considers approximations of the signal by sampling and quantization.

Section 6 derives the approximation bounds for filters corresponding to various approximations of the signal and presents a method of quantization of the observation process preserving the degree of the previous approximations.

2. MODEL AND FILTER

Throughout this work the following usual signal and observation model is considered.

The signal process is a q-dimensional continuous Markov diffusion defined by the stochastic differential equation

$$x_t = x_0 + \int_0^t f(s,x_s)ds + \int_0^t g(s,x_s)db_s \tag{2.1}$$

where b, called the state noise, is an r-dimensional Brownian motion, independent of x_0, and f and g are functions, sufficiently regular in order to guarantee the existence and uniqueness of the solution, [4], [6]. The observation process is a p-dimensional process given by

$$y_t = \int_0^t h(s,x_s)ds + w_t \tag{2.2}$$

where w, called the observation noise, is a p-dimensional Brownian motion and h, called the modulated signal, is supposed to satisfy the condition:

$$E\int_0^t |h(s,x_s)|^2 ds < \infty \tag{2.3}$$

for all $t < \infty$, condition under which the filtering equation was derived in [5].

In the general setting of the nonlinear filtering theory [5], b and w are correlated in such a way that (b,w) is a square integrable martingale. For structural reasons that will appear in the sequel, approximation schemes using the reference probability method, as in this work, are all proposed for independent state and observation noises. We are thus supposing that b and w are independent.

We denote by $|\cdot|$ the norm in $\mathbb{R}^n$, by v^i the i^{th} component of a vector v and by m^{ij} the entry (i,j) of a matrix m. $|m^{i\cdot}|$ stands for the norm of the vector $m^{i\cdot} = (m^{ij};\ j=1,2,\ldots)$. We also write $|m|^2 = \sum_i \sum_j (m^{ij})^2$. We simply denote by uv the scalar product of two elements $u,v \in \mathbb{R}^n$.

Since a filter is progressively computed in time, we may suppose that the time parameter of all the processes under consideration ranges over the finite interval [0,T].

Let G represent any one of the functions f and g. Then G is supposed to satisfy the following Lipschitz and linear growth conditions.

$$(2.4)\quad \begin{cases} |G(t,x) - G(t',x')| \leq K_0[|t-t'|(1+|x|+|x'|) + |x-x'|] \\ \\ |G(t,x)| \leq K_0(1+|x|) \end{cases}$$

for all $t,t' \in [0,T]$ and $x,x' \in \mathbb{R}^q$, where K_0 is a positive number not depending on t,t',x,x'.

x_0 is supposed to be a square integrable random variable, independent of b and w.

Let B (resp. Y) denote the space of all $\mathbb{R}^r$ (resp. $\mathbb{R}^p$)-valued continuous functions on [0,T] and denote by $\underline{\underline{B}}_T$ (resp. $\underline{\underline{Y}}_T$) its Borel σ-field under the sup-norm topology. $\underline{\underline{\mathbb{R}}}^q$ denotes the Borel σ-field of $\mathbb{R}^q$, and $\beta(\mathbb{R}^q)$, the space of all bounded Borel functions on $\mathbb{R}^q$.

P is the probability measure on $\underline{\underline{\mathbb{R}}}^q \times \underline{\underline{B}}_T \times \underline{\underline{Y}}_T =: \underline{\underline{F}}_T$ induced by (x_0,b,y). We suppose that all the probability spaces are complete. We put $\Omega = \mathbb{R}^q \times B \times Y$. Therefore, $(\Omega,\underline{\underline{F}}_T,P)$ is the probability space on which all the random processes under consideration are defined. We denote by $\underline{\underline{F}}_t$ (resp. $\underline{\underline{Y}}_t$) the sub-$\sigma$-fields generated by $\{(x_0,b_s,y_s);\ s \leq t\}$ (resp. $\{y_s;\ s \leq t\}$.

According to the Girsanov theorem [14], there is a probability measure P_0 on $\underline{\underline{F}}_T$, equivalent to P, under which y is a Brownian motion independent of (x_0,b) whose probability distribution, denoted by Q, remains unchanged. The corresponding Radon-Nikodym derivative is given by

$$\frac{dP}{dP_0} = \exp[\int_0^T C_s dy_s - \frac{1}{2}\int_0^T |C_s|^2 ds], \quad \text{with } C_s = h(s,x_s). \tag{2.5}$$

The probability law of y under P_0 is denoted by P_0^y. E(resp. E_0, E^Q, E_0^y) represents the expectation under P (resp. P_0, Q, P_0^y).

We put

$$Z_t = \exp[\int_0^t C_s dy_s - \frac{1}{2}\int_0^t |C_s|^2 ds]. \tag{2.6}$$

The process $Z = \{Z_t;\ t \in [0,T]\}$ is a martingale with respect to $\underline{\underline{F}} = \{\underline{\underline{F}}_t;\ t \in [0,T]\}$ and P_0, and

$$Z_t = E_0(Z_T | \underline{\underline{F}}_t) = \frac{dP_t}{dP_{0,t}} \quad \text{a.s.}$$

where P_t (resp. $P_{0,t}$) stands for the restriction to $\underline{\underline{F}}_t$ of P (resp. P_0).

If U is an $\underline{\underline{F}}_t$-measurable integrable random variable, then

$$E(U|\underline{\underline{Y}}_t) = \frac{E_0(Z_t\ U|\underline{\underline{Y}}_t)}{E_0(Z_t|\underline{\underline{Y}}_t)} \quad \text{and} \quad E_0(U|\underline{\underline{Y}}_t) = E^Q(U) \tag{2.7}$$

The second formula is a consequence of the independence of b and w.

A finite conditional measure σ_t can be defined on $\mathbb{R}^q$, such that, for all F $\in \beta(\mathbb{R}^q)$

$$\sigma_t(F) = E^Q[Z_t F(x_t)] = E_0[Z_t F(x_t)|\underline{\underline{Y}}_t]. \tag{2.8}$$

The measure-valued process σ is called the unnormalized filter (or filtering process) whereas the process π defined by the Kallianpur-Striebel formula [9]:

$$\pi_t\ (F) = \sigma_t(F)/\sigma_t(1) \tag{2.9}$$

is called the (normalized) filter or filtering process.

For a detailed presentation of the reference probability method we refer to [1] and [22]. We just mention here the fact that an application of the Itô differentiation rule to (2.8) to (2.9) successively leads to the Zakai equation [25] and to the nonlinear filtering equation, obtained in [5] by an application of the martingale representation theorem. In this paper we do not deal with these equations. The approximations we consider here concern formulas (2.8) and (2.9).

3. APPROXIMATIONS BY PERIODIC SAMPLING

3.1 FIRST APPROXIMATION

We suppose that the observation process y is sampled with a constant period of length δ and y is known by its samples $y_0 = 0$, $y_\delta, \dots, y_{n\delta}, \dots$ with $n \leq [T/\delta]$, ($[a/b]$ denotes the integer part of a/b). For a function F on $[0,T]$ we write $F_n^\delta = F(n\delta) - F((n-1)\delta)$ with $F_0^\delta = F_0$. We put $N = [T/\delta]$ and $y^\delta = \{y_n^\delta;\ n=0,1,\dots,N\}$. Under P_0, $\{y_n^{\delta,i};\ i=1,\dots,p;\ n = 1,\dots,N\}$ is a set of independent Gaussian random variables with means 0 and variances δ. We denote by $\underline{\underline{Y}}_t^\delta$ the sub-σ-field of $\underline{\underline{Y}}_t$ generated by $\{y_n^\delta;\ n \leq [t/\delta]\}$. The filtering process π^δ with respect to the sampled observation $\{y_{n\delta};\ n=0,\dots,N\}$ is given by

$$\pi_t^\delta(F) = E[F(x_t)|\underline{\underline{Y}}_t^\delta], \quad \text{for} \quad F \in \beta(\mathbb{R}^q). \tag{3.1}$$

As in (1.9) we have

$$\pi_t^\delta(F) = \sigma_t^\delta(F)/\sigma_t^\delta(1), \quad \text{for} \quad F \in \beta(\mathbb{R}^q) \tag{3.2}$$

where

$$\sigma_t^\delta(F) = E_0[Z_t F(x_t)|\underline{\underline{Y}}_t^\delta] = E_0^y[\sigma_t(F)|\underline{\underline{Y}}_t^\delta], \tag{3.3}$$

where σ^δ is the unnormalized filtering process corresponding to the sampled observation.

For $F \in \beta(\mathbb{R}^q)$ we have

$$\sigma_t^\delta(F) = E^Q[F(x_t)Z_t^\delta] \tag{3.4}$$

where

$$Z_t^\delta = \exp \sum_{n=0}^{[t/\delta]} (H_n^\delta y_n^\delta - \frac{\delta}{2}|H_n^\delta|^2) \tag{3.5}$$

with

$$H_n^\delta = \frac{1}{\delta}\int_{(n-1)\delta}^{n\delta} h(s,x_s)ds, \quad H_0^\delta = 0. \tag{3.6}$$

For a quick proof of these formulas, derived in [11], it is enough to notice that Z_t^δ is the ratio of the probability densities of $\{y_n^\delta;\ n \leq [t/\delta]\}$ under P and P_0, given x.

A recursive computation of σ^δ can be derived from (3.4) as shown in the following

THEOREM 3.1

Let us put

$$z_{n\delta} = \exp(H_n^\delta y_n^\delta - \frac{\delta}{2} |H_n^\delta|^2) \tag{3.7}$$

and define the conditional kernel $L_{n\delta}(x_{(n-1)\delta}, \cdot)$ on $\mathbb{R}^q$ by

$$\begin{aligned} L_{n\delta}(x_{(n-1)\delta}, F) &= \int_{\mathbb{R}^q} F(u) L_{n\delta}(x_{(n-1)\delta}, du) \\ &= E^Q[z_{n\delta} F(x_{n\delta}) | x_{(n-1)\delta}] \end{aligned} \tag{3.8}$$

for all $F \in \beta(\mathbb{R}^q)$. Then $\sigma_{n\delta}^\delta$; $n = 1,\dots,N$ satisfies the following recurrence equation:

$$\begin{aligned} \sigma_{n\delta}^\delta(F) &= \int_{\mathbb{R}^q} L_{n\delta}(u, F) \sigma_{(n-1)\delta}^\delta(du) \\ &= \sigma_{(n-1)\delta}^\delta[L_{n\delta}(\cdot, F)] \end{aligned} \tag{3.9}$$

for all $F \in \beta(\mathbb{R}^q)$; i.e.

$$\sigma_{n\delta}^\delta(dx) = \sigma_{(n-1)\delta}^\delta[L_{n\delta}(\cdot, dx)],$$

where σ_0^δ coincides with the law of x_0.

For $(n-1)\delta < t < n\delta$, σ_t^δ is given by

$$\begin{aligned} \sigma_t^\delta(F) &= \int_{\mathbb{R}^q} P_{(n-1)\delta, t}(u, F) \sigma_{(n-1)\delta}^\delta(du) \\ &= \sigma_{(n-1)\delta}^\delta[P_{(n-1)\delta, t}(\cdot, F)] \end{aligned} \tag{3.10}$$

for all $F \in \beta(\mathbb{R}^q)$, where $P_{s,t}$, $s < t$, is the conditional probability kernel of x_t given x_s. This formula can also be written as

$$\sigma_t^\delta(dx) = \sigma_{(n-1)\delta}^\delta[P_{(n-1)\delta, t}(\cdot, dx).]$$

Proof: For $t = n\delta$, formula (3.4) gives

$$\begin{aligned} \sigma_{n\delta}^\delta(F) &= E^Q[Z_{(n-1)\delta}^\delta \, z_{n\delta} \, F(x_{n\delta})] \\ &= E^Q\{Z_{(n-1)\delta}^\delta \, E^Q[z_{n\delta} F(x_{n\delta}) | x_{(n-1)\delta}]\} \\ &= E^Q\{Z_{(n-1)\delta}^\delta L_{n\delta}(x_{(n-1)\delta}, F)\} \end{aligned}$$

This is formula (3.9). For $(n-1)\delta < t < n\delta$, $Z_t^\delta = Z_{(n-1)\delta}^\delta$. Then (3.10) is obtained by

$$\sigma_t^\delta(F) = E^Q[Z^\delta_{(n-1)\delta}\, F(x_t)]$$
$$= E^Q\{Z^\delta_{(n-1)\delta}\, E^Q[F(x_t)\,|x_{(n-1)\delta}]\}. \qquad \square$$

Formula (3.9) allows the recursive determination of σ^δ at the sampling points and formula (3.10) gives it at any other point. The greatest difficulty in the computation of $\sigma^\delta_{n\delta}$, n=1,...,N, by (3.9), lies in the fact that the computation of the kernel $L_{n\delta}$ necessitates the complete knowledge of the probability law of x. This is very unrealistic, because there are few nonlinear diffusion processes whose laws can be handled. We therefore simplify the work by reducing the problem to the knowledge of the transition probabilities of x.

3.2 SECOND APPROXIMATION

We approximate H_n^δ by

(3.11) $$\tilde{H}_n^\delta = \frac{1}{\delta}\int_{(n-1)\delta}^{n\delta} h(s, x_{(n-1)\delta})ds$$

Then $z_{n\delta}$ depends only on $x_{(n-1)\delta}$. Let us denote it by $\tilde{z}_{n\delta}$. The kernel $L_{n\delta}$ becomes

(3.12) $$\tilde{L}_{n\delta}(x_{(n-1)\delta}, F) = \tilde{z}_{n\delta}\; P_{(n-1)\delta, n\delta}(x_{(n-1)\delta}, F)$$

for $F \in \beta(\mathbb{R}^q)$. We denote by $\tilde{\sigma}^\delta$ the unnormalized filter given by formulas (3.9) and (3.10), after $L_{n\delta}$ is replaced by $\tilde{L}_{n\delta}$, and by $\tilde{\pi}^\delta$ the corresponding normalized filter.

3.3 THIRD APPROXIMATION

Except for trivial cases the computation of the transition probabilities $P_{s,t}$ of a diffusion process is not an easy task and needs approximation procedures. One of the elementary approximation procedures lies in the Euler approximation of x defined by

(3.13) $$\hat{x}_{n\delta} = \hat{x}_{(n-1)\delta} + f[(n-1)\delta, \hat{x}_{(n-1)\delta}]\delta + g[(n-1)\delta, \hat{x}_{(n-1)\delta}]b_n^\delta$$

with $\hat{x}_0 = x_0$. The estimation of the approximation degree will be given in Section 5.

It is seen that $\{\hat{x}_{n\delta};\ n=0,1,\ldots,N\}$ is a Markov chain and its transition probabilities can be formulated in terms of the distributions of $b_{n\delta}$; $n=1,2,\ldots,N$.

We approximate x by

$$\hat{x}_t = \sum_{n=0}^{N} \hat{x}_{n\delta} 1_{[n\delta,(n+1)\delta[}(t). \tag{3.14}$$

We then replace x by $\hat{x}$ in formulas (3.4) to (3.9) and obtain by (3.9) the corresponding unnormalized filter $\hat{\sigma}^{\delta}_{n\delta}$, $n=0,1,\ldots,N$. We put $\hat{\sigma}^{\delta}_t = \hat{\sigma}^{\delta}_{n\delta}$ for $n\delta < t < (n+1)\delta$. $\hat{\sigma}^{\delta}$ is an approximation of σ. We denote by $\hat{\pi}^{\delta}$ the normalization of $\hat{\sigma}^{\delta}$.

In Section 5 we shall consider another approximation where $\hat{x}_{n\delta}$ will be replaced by a finite space valued random variable. Before going further into the approximation procedures, we are going to develop, in the next Section, the general method of estimation of approximation degrees and apply it to the first and second approximations which have their own theoretical interest.

4. ESTIMATION OF APPROXIMATION DEGREES.

Suppose $\mathring{\pi}$ represents any approximation of π corresponding to a periodic sampling of y with period δ. We are interested in the asymptotic behavior of $\mathring{\pi}(F)$, for $F \in \beta(\mathbb{R}^q)$; more precisely, in proving the L^1-convergence of $\mathring{\pi}(F)$ to $\pi(F)$ and estimating the speed of convergence as $\delta \to 0$.

The L^p-norm under P (resp. P_0) is denoted by $||\cdot||_p$ (resp. $||\cdot||_{0,p}$). For an $\mathbb{R}^n$-valued random variable U, the norm $||U||_p$ is defined by $||U||_p^p = E\{[\sum_{i=1}^{n} (U^i)^2]^{p/2}\}$.

In all the approximation schemes considered here, $C_s = h(s,x_s)$, $s<t$, is approximated by a left continuous step process which is constant on sampling intervals. We denote by $\mathring{C}_{t,s}$ the process coinciding with this approximating step process on $[0,[t/\delta]\delta]$ and vanishing outside of this interval. We put $\mathring{C}_{t,s} = 0$ for $t<\delta$. Notice that $\mathring{C}_{T,s} = \mathring{C}_{t,s}$ for $s \leq [t/\delta]\delta$. The corresponding likelihood ratio formula can then be written as follows:

$$\mathring{Z}_t = \exp\left(\int_0^t \mathring{C}_{t,s} dy_s - \frac{1}{2}\int_0^t |\mathring{C}_{t,s}|^2 ds\right). \tag{4.1}$$

For the first (resp. second) approximation, we have

$$\mathring{C}_{t,s} = H_n^\delta \text{ (resp. } \tilde{H}_n^\delta) \quad \text{for } (n-1)\delta < s \leq n\delta \leq t, \tag{4.2}$$

and put

$$C_{t,s}^\delta \text{ (resp. } \tilde{C}_{t,s}^\delta) := \mathring{C}_{t,s}. \tag{4.3}$$

In some other approximation schemes, as in the third approximation, x is replaced by another process $\mathring{x}$ approximating it. Particular notations corresponding to these schemes will be introduced when we shall be considering them.

In the case where the likelihood ratio Z is approximated by $\mathring{Z}$, the filters σ and π are approximated by $\mathring{\sigma}$ and $\mathring{\pi}$, respectively, defined by

$$\mathring{\sigma}_t(F) = E^Q[\mathring{Z}_t F(\mathring{x}_t)] \text{ and } \mathring{\pi}_t(F) = \mathring{\sigma}_t(F)/\mathring{\sigma}_t(1) \tag{4.4}$$

for $F \in \beta(\mathbb{R}^q)$.

At the first step we are going to derive a bound for $||\pi(F) - \mathring{\pi}(F)||_1$ under the hypothesis that

$$\begin{cases} \{C_t;\ t \in [0,T]\} \text{ and } \{\mathring{C}_{T,t};\ t\in[0,T]\} \text{ belong to} \\ L^2([0,T] \times \Omega,\ \underline{\underline{T}} \oplus \underline{\underline{F}}_T,\ dt \times dP) \end{cases} \tag{4.5}$$

where $\underline{\underline{T}}$ is the Borel σ-field of [0,T].

For this purpose the following Girsanov theorem constitutes a useful tool. We write it as a lemma.

LEMMA 4.1

Under the probability $\mathring{P}$, defined by $d\mathring{P} = \mathring{Z}_T dP_0$, the process

$$\mathring{w}_t = y_t - \int_0^t \mathring{C}_{T,s} ds \tag{4.6}$$

is a Brownian motion independent of (x_o, b) and the law of (x_o, b) is always Q.□

The mathematical expectation under $\mathring{P}$ is denoted by $\mathring{E}$. We choose $F \in \beta(\mathbb{R}^q)$ once and for all and write

$$E^Q(Z_t)[\pi(F) - \overset{\circ}{\pi}(F)] = E^Q[Z_tF(x_t] - \frac{E^Q[\overset{\circ}{Z}_tF(\overset{\circ}{x}_t)]}{E^Q(\overset{\circ}{Z}_t)} E^Q(Z_t) =$$

$$= E^Q[Z_tF(x_t) - \overset{\circ}{Z}_tF(\overset{\circ}{x}_t)] - \frac{E^Q[\overset{\circ}{Z}_tF(\overset{\circ}{x}_t)]}{E^Q(\overset{\circ}{Z}_t)}[E^Q(Z_t) - E^Q(\overset{\circ}{Z}_t)].$$

Therefore,

(4.7) $||\pi_t(F) - \overset{\circ}{\pi}_t(F)||_1 \le ||\sigma_t(F) - \overset{\circ}{\sigma}_t(F)||_{0,1} + ||\overset{\circ}{\pi}_t(F)E^Q(Z_t-\overset{\circ}{Z}_t)||_{0,1}$

We remark that $\overset{\circ}{\pi}$ is a regular conditional probability measure and that $|\overset{\circ}{\pi}(F)| \le ||F||$, for almost all trajectories of y, where $||F||$ denotes the sup-norm of F. Thus

(4.8) $$||\overset{\circ}{\pi}(F)E^Q(Z_t-\overset{\circ}{Z}_t)||_{0,1} \le ||F|| \; ||Z_t-\overset{\circ}{Z}_t||_{0,1}$$

On the other hand, we can write

(4.9)
$$\begin{aligned} &||\sigma_t(F) - \overset{\circ}{\sigma}_t(F)||_{0,1} \le \\ &\le ||E^Q[(Z_t-\overset{\circ}{Z}_t)F(x_t)]||_{0,1} + ||E^Q[\overset{\circ}{Z}_t(F(x_t) - F(\overset{\circ}{x}_t))]||_{0,1} \\ &\le ||(Z_t - \overset{\circ}{Z}_t)F(x_t)||_{0,1} + ||\overset{\circ}{Z}_t[F(x_t) - F(\overset{\circ}{x}_t)]||_{0,1} \\ &\le ||F|| \; ||Z_t-\overset{\circ}{Z}_t||_{0,1} + \overset{\circ}{E}|F(x_t) - F(\overset{\circ}{x}_t)| = \\ &||F|| \; ||Z_t-\overset{\circ}{Z}_t||_{0,1} + ||F(x_t) - F(\overset{\circ}{x}_t)||_1 \end{aligned}$$

where we used the statement of Lemma 4.1.

From inequalities (4.7), (4.8) and (4.9) we finally deduce the following

(4.10) $||\pi_t(F) - \overset{\circ}{\pi}_t(F)||_1 \le 2||F|| \; ||Z_t-\overset{\circ}{Z}_t||_{0,1} + ||F(x_t)-F(\overset{\circ}{x}_t)||_1$.

Next, we derive a bound for $||Z_t-\overset{\circ}{Z}_t||_{0,1}$.

By using the inequality $|e^x-e^y| \le |x-y|(e^x+e^y)$, $\forall x,y \in \mathbb{R}$, we can write

(4.11) $$|Z_t-\overset{\circ}{Z}_t| \le (Z_t+\overset{\circ}{Z}_t)|U_t| = Z_t|U_t| + \overset{\circ}{Z}|U_t|$$

where

(4.12) $$U_t = \int_0^t (C_s-\overset{\circ}{C}_{t,s})dy_s - \frac{1}{2}\int_0^t(|C_s|^2-|\overset{\circ}{C}_{t,s}|^2)ds.$$

We have, by (2.2)

(4.13)
$$\begin{aligned} &E_0(Z_t|U_t|) = E(|U_t|) = \\ &= E(|\int_0^t(C_s-\overset{\circ}{C}_{t,s})dw_s + \frac{1}{2}\int_0^t|C_s-\overset{\circ}{C}_{t,s}|^2ds|) \\ &\le (\int_0^tE|C_s-\overset{\circ}{C}_{t,s}|^2ds)^{1/2} + \frac{1}{2}\int_0^t E|C_s-\overset{\circ}{C}_{t,s}|^2ds \end{aligned}$$

According to Lemma 2.1 we can write

(4.14) $$E_0[\hat{Z}_t|U_t|] \le E_0(\hat{Z}_t|\int_0^t(C_s-\hat{C}_{t,s})dy_s|) + \frac{1}{2}\int_0^t E||C_s|^2-|\hat{C}_{t,s}|^2|ds$$

For the evaluation of the first term of the right hand side, we remark that, for fixed x and $\mathring{x}$, $\hat{Z}_t$ is measurable with respect to $\underline{\underline{Y}}_t^\delta$. We can therefore write

(4.15) $$E_0(\hat{Z}_t|\int_0^t(C_s-\hat{C}_{t,s})dy_s|) = E_0[\hat{Z}_t E_0^y(|\int_0^t(C_s-\hat{C}_{t,s})dy_s||\underline{\underline{Y}}_t^\delta)] \le$$
$$\le E_0\{\hat{Z}_t[E_0^y(|\int_0^t(C_s-\hat{C}_{t,s})dy_s|^2|\underline{\underline{Y}}_t^\delta)]^{1/2}\}$$

For fixed x and $\mathring{x}$, the stochastic integral of this formula is a centered Gaussian random variable and we need to calculate its conditional second moment with respect to a σ-field generated by a finite number of centered Gaussian random variables. Therefore the conditional expectation of the stochastic integral term is equal to its projection onto the Gaussian space generated by $y_1^\delta,\dots,y_{n\delta}^\delta$, with $n=[t/\delta]$. It is easily seen that

$$E_0^y\{[\int_0^t(C_s-\hat{C}_{t,s})dy_s]|\underline{\underline{Y}}_t^\delta\} = \int_0^t(C_{t,s}^\delta - \hat{C}_{t,s})dy_s.$$

Since (always for fixed x and $\mathring{x}$)

$$\int_0^t(C_s-\hat{C}_{t,s})dy_s - \int_0^t(C_{t,s}^\delta - \hat{C}_{t,s})dy_s = \int_0^t(C_s - C_{t,s}^\delta)dy_s$$

is independent of $\underline{\underline{Y}}_t^\delta$, we see that the conditional variance of $\int_0^t(C_s - C_{t,s})dy_s$ is $\int_0^t|C_s - C_{t,s}^\delta|^2ds$. Consequently, we have

(4.16) $$E_0^y(|\int_0^t(C_s - \hat{C}_{t,s})dy_t|^2|\underline{\underline{Y}}_t^\delta) = = [\int_0^t(C_{t,s}^\delta - \hat{C}_{t,s})dy_s]^2 + \int_0^t|C_s - C_{t,s}^\delta|^2ds.$$

By using again Lemma 4.1, from (4.15) and (4.16) we derive:

(4.17) $$E_0(\hat{Z}_t|\int_0^t(C_s-\hat{C}_{t,s})dy_s|) \le \le \hat{E}|\int_0^t(C_{t,s}^\delta - \hat{C}_{t,s})dy_s| + E(\int_0^t|C_s - C_{t,s}^\delta|^2ds)^{1/2}.$$

By Lemma 4.1 again, we have

$$(4.18) \qquad \hat{E}|\int_0^t (C^\delta_{t,s} - \hat{C}_{t,s}) dy_s| \le$$
$$\le \hat{E}|\int_0^t (C^\delta_{t,s} - \hat{C}_{t,s})\hat{C}_{t,s} ds| + \hat{E}|\int_0^t (C^\delta_{t,s} - \hat{C}_{t,s}) d\mathring{w}_s|$$
$$\le E|\int_0^t (C^\delta_{t,s} - \hat{C}_{t,s})\hat{C}_{t,s} ds| + [\int_0^t E|C^\delta_{t,s} - \hat{C}_{t,s}|^2 ds]^{1/2}.$$

According to (4.16), (4.17) and (4.18), inequality (4.14) gives:

$$(4.19) \qquad E_0[\hat{Z}_t |U_t|] \le$$
$$\le E|\int_0^t (C^\delta_{t,s} - \hat{C}_{t,s})\hat{C}_{t,s} ds| + (\int_0^t E|C^\delta_{t,s} - \hat{C}_{t,s}|^2 ds)^{1/2} +$$
$$+ E(\int_0^t |C_s - C^\delta_{t,s}|^2 ds)^{1/2} + \tfrac{1}{2}\int_0^t E||C_s|^2 - |\hat{C}_{t,s}|^2| ds$$

Inequalities (4.13) and (4.19) provide a bound for (4.11).

Let us denote by $\| \cdot \|_t$ the norm in $L^2([0,t] \times \Omega)$ where the measure is $dt \times dP$. Then we can express the main tool which will allow the evaluation of the approximation degrees for filters.

MAIN LEMMA 4.2

Under Hypothesis (4.5), the following inequality holds.

$$(4.20) \qquad ||Z_t - \hat{Z}_t||_{0,1} \le$$
$$\le \|C - C^\delta_t\|_t + \|C - \hat{C}_t\|_t (1 + \tfrac{1}{2}\|C - \hat{C}_t\|_t + \tfrac{1}{2}\|C + \hat{C}_t\|_t) +$$
$$+ \|C^\delta_t - \hat{C}_t\|_t (1 + \|\hat{C}_t\|_t). \qquad \square$$

Now, we are able to express, in terms of δ, a bound for (4.10), under the following hypothesis.

$$(4.21) \qquad \begin{cases} |h(t,x) - h(t',x')| \le K_0[|t-t'|(1+|x|+|x'|)+|x-x'|] \\ |h(t,x)| \le K_0(1 + |x|). \end{cases}$$

where K_0 is a positive number not depending on t, t', x and x'. We could choose K_0 different from the one used in (2.4) for f and g. But this would render the notations more complicated.

We first recall the following classical result, (cf. Section 5).

There exist positive constants K_1 and K_2 such that

$$(4.22) \qquad ||x_t||_2 \le K_1 \quad \text{and} \quad ||x_t - x_s||_2 \le K_2|t-s|^{1/2}.$$

These inequalities are only consequences of the square integrability of x_0

and of conditions (2.4) for f and g.

In this section we only consider the two cases of the first and second approximations. In these two cases $\overset{\circ}{x} = x$. Inequalities (4.21) and (4.22) guarantee that Hypothesis (4.5) is fulfilled. Therefore, we can apply Lemma 4.2.

For $\overset{\circ}{\pi} = \pi^{\delta}$, (4.10) and (4.20) give

$$(4.23) \qquad ||\pi_t(F) - \pi_t^{\delta}(F)||_1 \le$$
$$\le 2\|F\| \, \|C-C_t^{\delta}\|_t \, (2 + \tfrac{1}{2}\|C-C_t^{\delta}\|_t + \tfrac{1}{2}\|C+C_t^{\delta}\|_t)$$

By putting $t'=[t/\delta]\delta$, we can write

$$\int_0^t E|C_s - C_{t,s}^{\delta}|^2 ds =$$
$$= \int_{t'}^{t} E|C_s|^2 ds + \int_0^{t'} E|C_s - C_{t,s}^{\delta}|^2 ds \le$$
$$\le \delta \sup_s ||C_s||_2^2 + t' \sup_{s \le t} ||C_s - C_{t,s}^{\delta}||_2^2.$$

(4.21) and (4.22) give, for $(n-1)\delta < s \le n\delta \le t$,

$$(4.24) \qquad ||C_s - C_{t,s}^{\delta}||_2 \le ||h(s,x_s) - \tfrac{1}{\delta}\int_{(n-1)\delta}^{n\delta} h(u,x_u)du||_2 =$$
$$= \tfrac{1}{\delta}||\int_{(n-1)\delta}^{n\delta}[h(s,x_s)-h(u,x_u)]du||_2 \le$$
$$\le K_0\{\tfrac{1}{\delta}\int_{(n-1)\delta}^{n\delta} E[|u-s|(1+|x_u|+|x_s|) + |x_u - x_s|]^2 du\}^{1/2} \le$$
$$\le K_0 \, [\delta(1+2K_1) + K_2\sqrt{\delta}].$$

As $||C_s||_2 \le K_0(1+K_1)$, we can write

$$(4.25) \qquad \|C-C_t^{\delta}\|_t \le K_0\{1+K_1 + \sqrt{T}\,[K_2 + (1+2K_1)\sqrt{\delta}]\}\sqrt{\delta} =: A_1\sqrt{\delta}.$$

We also have $||C_{t,s}^{\delta}||_2 \le K_0(1+K_1)$. Therefore, we can formulate the following result.

THEOREM 4.3

Under hypotheses (2.4), (4.21) and $x_0 \in L^2(\Omega, \underline{\underline{F}}_T, P)$ we have

$$(4.26) \qquad ||\pi_t(F) - \pi_t^{\delta}(F)||_1 \le 2||F||A^{\delta}\sqrt{\delta}$$

where

$$(4.27) \qquad A^{\delta} = A_1[2+\tfrac{1}{2}A_1\sqrt{\delta} + \sqrt{T}\,K_0(1+K_1)]$$

with A_1 defined by (4.25). □

REMARK 4.4

Inequality (4.26) holds also for $\tilde{\pi}^{\delta}$, corresponding to the second approximation. The only difference in the proof is the fact that in (4.24) x_u is to be replaced by $x_{(n-1)\delta}$.

REMARK 4.5.

The equality $\underline{\underline{Y}}_t = \bigvee_{\delta} \underline{\underline{Y}}_t^{\delta}$ implies that, for $F \in \beta(\mathbb{R}^q)$, $\pi^{\delta}(F)=E[F(x_t)|\underline{\underline{Y}}_t^{\delta}]$ converges to $\pi(F)=E[F(x_t)|\underline{\underline{Y}}_t]$ in all L^m, $m \geq 1$, [17]. The convergence is even a.s. if the sets of sample points $\{0,\delta,\ldots,N\delta\}$ form an increasing sequence as $\delta \to 0$.

5. APPROXIMATIONS OF THE SIGNAL PROCESS

After a study of the Euler approximation in L^k, we give a quantization scheme preserving the degree of approximation. First of all we are going to give an estimation of the constants $K_1(k)$ and $K_2(k)$ such that inqualities

(5.1) $$||x_t||_k \leq K_1(k) \quad \text{and} \quad ||x_t - x_s||_k \leq K_2(k)\ |t-s|^{1/2}$$

hold when $x_0 \in L^k$. For k=2, we put $K_1(2)=K_1$ and $K_2(2)=K_2$.

In current literature ([4], [6], [14]) these constants are determined for even k by using the Itô calculus. But one may need to consider the above inequalities for arbitrary $k \geq 2$. We give here a rapid estimation of K_1 and K_2 by using the Burkholder-Davis-Gundy inequality, [2].

Suppose $x_0 \in L^k$, $k \geq 2$, and let N be a positive number and define $\tau=\inf\{t; |x_t| \geq N\}$. Since x is continuous, and $|x_0|<\infty$ a.s., $\tau \neq 0$ for sufficiently large N and $\tau \to T$ a.s. as $N \to \infty$. We put $t\wedge\tau = \min(t,\tau)$. In order to simplify the notations we write $||\cdot||$ instead of $||\cdot||_k$.

We have

(5.2) $$x_{t\wedge\tau} = x_0 + \int_0^t 1_{[0,\tau]}(s) f(s,x_s)ds + \int_0^t 1_{[0,\tau]}(s) g(s,x_s)db_s.$$

$$||x_{t\wedge\tau}|| \leq ||x_0|| + q^{\frac{1}{2}-\frac{1}{k}}\, t^{1-\frac{1}{k}}\Big[\sum_{i=1}^{q} \int_0^t E(1_{[0,\tau]}(s)|f^i(s,x_s)|^k)ds\Big]^{\frac{1}{k}} +$$

$$+4k(qt)^{\frac{1}{2}-\frac{1}{k}}\Big[\sum_{i=1}^{q}\int_0^t E(1_{[0,\tau]}(s)|g^{i\cdot}(s,x_s)|^k)ds\Big]^{\frac{1}{k}}$$

where $g^{i\cdot}$ represents the vector $(g^{i1},\dots,g^{ir})$.

For k=2, the coefficient 4k can be replaced by 1. By using (2.4) we can write

$$||x_{t\wedge\tau}|| \le ||x_0|| + K_0(qt)^{\frac{1}{2}}(t^{\frac{1}{2}}+4k)t^{-\frac{1}{k}}[\int_0^t E(1+|x_{s\wedge\tau}|)^k ds]^{\frac{1}{k}}.$$

Therefore

$$||x_{t\wedge\tau}||^k \le 2^{k-1}||x_0||^k + 2^{k-1}K_0^k(qt)^{\frac{k}{2}}(t^{\frac{1}{2}}+4k)^k\, t^{-1}\int_0^t 2^{k-1}(1+||x_{s\wedge\tau}||^k)ds$$

$$\le 2^{k-1}||x_0||^k + 2^{2k-2}K_0^k(qT)^{\frac{k}{2}}(T^{\frac{1}{2}}+4k)^k +$$

$$+ 2^{2k-2}K_o^k(qT)^{\frac{k}{2}}(T^{\frac{1}{2}}+4k)^k T^{-1}\int_0^t ||x_{s\wedge\tau}||^k ds.$$

Then the Gronwall lemma gives

$$(5.3)\qquad \begin{cases} ||x_{t\wedge\tau}||^k \le A \exp B \\ \text{where} \\ A = 2^{k-1}||x_0||^k + B \\ B = 2^{2k-2}K_0^k(qT)^{\frac{k}{2}}(T^{\frac{1}{2}}+4k)^k \end{cases}.$$

Since the right hand side of the last inequality does not depend on N, we can let N go to infinity and obtain

$$(5.4)\qquad ||x_t||_k \le A^{\frac{1}{k}}\exp(B/k) =: K_1(k)$$

$K_2(k)$ can be obtained in terms of $K_1(k)$, starting from

$$||x_t-x_s|| \le q^{\frac{1}{2}-\frac{1}{k}}(t-s)^{1-\frac{1}{k}}\Big[\sum_{i=1}^{q}\int_s^t E(|f^i(u,x_u)|^k)du\Big]^{\frac{1}{k}} +$$

$$+ 4k[q(t-s)]^{\frac{1}{2}-\frac{1}{k}}\Big[\sum_{i=1}^{q}\int_s^t E(|g^{i\cdot}(u,x_u)|^k)du\Big]^{\frac{1}{k}},$$

with t>s, and by using the second inequality (2.4) for f^i and $g^{i\cdot}$ and (5.4).

For various approximations of x we refer to [15], [18] and [21]. We just derive here a few inequalities concerning the Euler approximation $\hat{x}$, defined by

(3.13) and (3.14). We first determine a bound for $||\hat{x}_t||_k$, under the hypothesis that $x_0 \in L^k$, $k \geq 2$.

Let us define $\hat{f}$ and $\hat{g}$ as follows

(5.5) $\quad \hat{f}(t,\hat{x}_t) = f(n\delta,\hat{x}_{n\delta}),\ \hat{g}(t,\hat{x}_t) = g(n\delta,\ \hat{x}_{n\delta}),$ for $n\delta \leq t < (n+1)\delta$.

Then we can write

(5.6) $$\hat{x}_{n\delta} = x_0 + \int_0^{n\delta} \hat{f}(s,\hat{x}_s)(s,\hat{x}_s)ds + \int_0^{n\delta} \hat{g}(s,\hat{x}_s)db_s$$

and use the method that allowed the passage from (5.1) to (5.4) to find that $||\hat{x}_{n\delta}||_k$ also is bounded by $K_1(k)$ given in (5.4).

The same method can be applied for an estimation of $||x_{n\delta}-\hat{x}_{n\delta}||_x$. We have for $t = n\delta$

$$||x_t-\hat{x}_t|| \leq K_0(qT)^{\frac{1}{2}} (T^{\frac{1}{2}} + 4k)T^{-\frac{1}{k}}\{\sum_{i=0}^{n-1}\int_{i\delta}^{(i+1)\delta}E[(s-i\delta)(1+|x_s|+|\hat{x}_{i\delta}|) + + |x_s - x_{i\delta}| + |x_{i\delta} - \hat{x}_{i\delta}|]^k ds\}^{\frac{1}{k}}$$

and

$$||x_t-\hat{x}_t||^k \leq [K_0(qT)^{\frac{1}{2}} (T^{\frac{1}{2}} + 4k)]^k\, T^{-1}\, 2^{k-1}\, \{T[\delta(1+2K_1(k)) + K_2(k)\sqrt{\delta}]^k + \delta \sum_{i=0}^{n-1} ||x_{i\delta}-\hat{x}_{i\delta}||^k\}.$$

By putting $u_s = ||x_{i\delta}-\hat{x}_{i\delta}||^k$ for $i\delta \leq s < (i+1)\delta$, the sum in the above inequality, multiplied by δ can be written as $\int_0^t u_s ds$. Therefore, the Gronwall lemma can be applied and gives the following inequality.

$$||x_{n\delta} - \hat{x}_{n\delta}||_k \leq (\hat{A} \exp \hat{B})\sqrt{\delta}, \qquad \text{where}$$

(5.7) $$\begin{cases} \hat{A} = (k\hat{B})^{\frac{1}{k}}\,[(1+2K_1(k))\sqrt{\delta} + K_2(k)] \\ \hat{B} = \frac{1}{k}[2^{1-\frac{1}{k}}\, K_0(qT)^{\frac{1}{2}}\, (T^{\frac{1}{2}} + 4k)]^k \end{cases}$$

Finally, for arbitrary $t \in\,]n\delta,\ (n+1)\delta[$, we can write

(5.8) $$||x_t - \hat{x}_t||_k \leq [\hat{A} \exp \hat{B} + K_2(k)]\sqrt{\delta}$$

We define

(5.9) $$\hat{K}(k) := \hat{A} \exp \hat{B}.$$

Next, we proceed to the quantization of $\hat{x}$.

We fix k once and for all and suppose that x_0 has a continuous distribution function with finite k^{th} absolute moment.

In order to simplify the notation we write Equation (3.13) as follows:

(5.10) $$X_n = X_{n-1} + \hat{f}(n-1,X_{n-1}) + \hat{g}(n-1,\ X_{n-1})\ \sqrt{\delta}\ b_n, \quad n=0,1,\ldots,N$$

where $\{b_n^i,\ n=0,1,\ldots,N,\ i=1,\ldots,q\}$ is a set of independent normalized Gaussian random variables and X_n, $\hat{f}(n-1,X_{n-1})$, $\hat{g}(n-1,X_{n-1})$ stand for $\hat{x}_{n\delta}$, $f((n-1)\delta,\ \hat{x}_{(n-1)\delta})$, $g((n-1)\delta,\ \hat{x}_{(n-1)\delta})$, respectively.

We choose a finite increasing sequence $u_0,u_1,\ldots,u_I$ in $\mathbb{R}$ and real coefficients $a_0,a_1,\ldots,a_{I-1}$ in such a way that if $B = \sum_{i=0}^{I-1} a_i 1_{]u_i,u_{i+1}]}$ and if x is a real random variable with distribution N(0,1), we have $||x-B(x)||_k \le \gamma$. We put $\bar{b}_n^i = B(b_n^i)$ and $a = \sup_k |a_k|$. We approximate x_0 by a finite space valued random variable $\bar{x}$ in order to have $||x_0-\bar{x}_0||_k \le \gamma$ and denote $\alpha_0 = \sup_{i,\omega} |\bar{X}_0^i(\omega)|$.

We approximate (X_n) by a finite space valued Markov chain $(\bar{X}_n)$ defined by

(5.11) $$\bar{X}_n = \bar{X}_{n-1} + \bar{f}(n-1,\ \bar{X}_{n-1})\delta + \bar{g}(n-1,\ \bar{X}_{n-1})\ \sqrt{\delta}\ \bar{b}_n.$$

The components of $\bar{X}_n$ take a finite number of values in an interval $[-\alpha,\alpha]$. The number α will be determined in the sequel. Functions $\hat{f}^i(n,\cdot)$ and $\hat{g}^{ij}(n,\cdot)$ are approximated by step functions $\bar{f}^i(n,\cdot)$ and $\bar{g}^{ij}(n,\cdot)$ on $[-\alpha,\alpha]$, such that

$$\sup_{i,n,x} |\hat{f}^i(n,x) - \bar{f}^i(n,x)| \le \eta \quad \text{and} \quad \sup_{i,j,n,x} |\hat{g}^{ij}(n,x) - \bar{g}^{ij}(n,x)| \le \eta,$$

(the components of x are restricted to $[-\alpha,\ \alpha]$). The values of δ, $\sqrt{\delta}$, x_0^i, $\bar{b}_n^i$, $\bar{f}^i$, $\bar{g}^{ij}$ are chosen in a finite periodic subset of $[-\alpha,\alpha]$.

By taking the difference of the two equations (5.10) and (5.11) we can write

$$||X_n-\bar{X}_n|| \le ||X_{n-1}-\bar{X}_{n-1}|| + ||\hat{f}(n-1,X_{n-1}) - \hat{f}(n-1,\ \bar{X}_{n-1})||\delta +$$
$$+ ||\hat{f}(n-1,\ \bar{X}_{n-1}) - \bar{f}(n-1,\ \bar{X}_{n-1})||\delta + ||\hat{g}(n-1,\ X_{n-1})(b_n-\bar{b}_n)||\sqrt{\delta}$$

$$+ \|[\hat{g}(n-1,X_{n-1}) - \hat{g}(n-1,\bar{X}_{n-1})]\bar{b}_n\| \sqrt{\delta} +$$

$$+ \|[\hat{g}(n-1, \bar{X}_{n-1}) - \bar{g}(n-1,\bar{X}_{n-1})]\bar{b}_n\|\sqrt{\delta} \le$$

$$\le \|X_{n-1} - \bar{X}_{n-1}\| + K_0\|X_{n-1} - \bar{X}_{n-1}\|\delta + \sqrt{q}\eta\delta +$$

$$+ \sqrt{q}r\ K_0(1+K_1(k))\gamma\sqrt{\delta} + \sqrt{q}rK_0\|X_{n-1} - \bar{X}_{n-1}\|(M(k)+\gamma)\sqrt{\delta} +$$

$$+ \sqrt{q}r\eta\ (M(k) + \gamma)\ \sqrt{\delta}\quad .$$

where $M(k)$ is the L^k-norm of a random variable with distribution $N(0,1)$ and we denoted $\|\cdot\|_k$ simply by $\|\cdot\|$ and used the inequality $\|\bar{b}_n\|_k \le \|b_n\|_k + \|\bar{b}_n - b_n\|_k$.

We can write

$$(5.12)\qquad \begin{cases} \|X_n - \bar{X}_n\| \le u\|X_{n-1} - \bar{X}_{n-1}\| + v \\ \text{where} \\ u = 1 + \quad K_0\delta + \sqrt{q}rK_0(M(k) + \gamma)\sqrt{\delta} \\ v = \sqrt{q}\ \sqrt{\delta}\ [\eta\sqrt{\delta} + rK_0(1 + K_1(k))\gamma + r(M(k) + \gamma)\eta] \end{cases}$$

Successive iterations of this inequality give

$$(5.13)\qquad \|X_n - \bar{X}_n\|_k \le v\,\frac{u^n - 1}{u - 1} + u^n\gamma$$

We want $\|X_n - \bar{X}_n\|_k$ to be of the order of $\sqrt{\delta}$. Therefore, we can choose

$$(5.14)\qquad \eta = v_1\ u^{-N}\ \sqrt{\delta},\ \gamma = v_2 u^{-N}\sqrt{\delta}$$

where $N = [T/\delta]$ and v_1 and v_2 are constants that can be chosen independently of δ.

Denoting by $\bar{K}(k)\sqrt{\delta}$ the right hand side of (5.13) when η and γ are replaced by (5.14) we write

$$(5.15)\qquad \|X_n - \bar{X}_n\|_k \le \bar{K}(k)\sqrt{\delta}\quad .$$

The approximation of $\hat{f}(n,\cdot)$ and $\hat{g}(n,\cdot)$ by step functions on $[-\alpha,\alpha]$ depends, of course, on the magnitude of α. Therefore we need to determine a lower bound for it. Equation (5.11), can be written for the i^{th} component as follows:

$$\bar{X}^i_n = \bar{X}^i_{n-1} + \bar{f}^i(n-1,\ \bar{X}_{n-1})\delta + \sum_{j=1}^{r} \bar{g}^{ij}(n-1,\bar{X}_{n-1})\ \sqrt{\delta}\ \bar{b}^j_n$$

$$= \bar{X}^i_{n-1} + [\bar{f}^i(n-1,\bar{X}_{n-1}) - \hat{f}^i(n-1,\bar{X}_{n-1})]\delta + \hat{f}^i(n-1,\bar{X}_{n-1})\delta +$$

$$+ \sum_{j=1}^{r} [\bar{g}^{ij}(n-1,\bar{X}_{n-1}) - \hat{g}^{ij}(n-1,\bar{X}_{n-1})]\sqrt{\delta}\ \bar{b}^j_n + \sum_{j=1}^{r} \hat{g}^{ij}(n-1,\bar{X}_{n-1})\sqrt{\delta}\ \bar{b}^j_n$$

From this we deduce,

$$|\bar{X}^i_n| \le |\bar{X}^i_{n-1}| + \eta\delta + K_0(1+|\bar{X}_{n-1}|)\delta + ra\eta\sqrt{\delta} + rK_0a(1+|\bar{X}_{n-1}|)\ \sqrt{\delta}$$

Let us put $\alpha_n = \sup\limits_{i,\omega} |\bar{X}^i_n(\omega)|$. Then we can write

$$\alpha_n \le \alpha_{n-1} + \eta\delta + K_0(1 + \sqrt{q}\ \alpha_{n-1})\delta + ra\eta\sqrt{\delta} + rK_0a(1+\sqrt{q}\ \alpha_{n-1})\sqrt{\delta}$$

$$\alpha_n \le \alpha_{n-1}(1+K_0\sqrt{q}\sqrt{\delta}(ra + \sqrt{\delta})) + \sqrt{\delta}(\eta + K_0)(ra + \sqrt{\delta})$$

By iterating successively we get the following inequality for α_n:

$$\alpha_n \le \left[\frac{\eta + K_0}{K_0\sqrt{q}} + \alpha_0\right] [1 + K_0\sqrt{q}\ \sqrt{\delta}(ra + \sqrt{\delta})]^N - \frac{\eta + K_0}{K_0\sqrt{q}} =: \alpha \tag{5.16}$$

where $N = [T/\delta]$.

We see that once δ is given, η and γ can be chosen according to (5.14). The choice of γ imposes that of a, and finally α can be chosen to be equal to the right hand side of (5.16).

Suppose that we approximate $\hat{x}_{n\delta} = X_n$ by $\bar{x}_{n\delta} = \bar{X}_n$ as above and we define $\bar{x}_t$ by

$$\bar{x}_t = \bar{x}_{n\delta} \quad \text{for} \quad n\delta \le t < (n+1)\delta. \tag{5.17}$$

We obviously have $||\hat{x}_t - \bar{x}_t||_k \le \bar{K}(k)\sqrt{\delta}$, (cf. (5.15)). By replacing $\hat{x}$ by $\bar{x}$ in the third approximation we get new unnormalized and normalized filters that we denote by $\bar{\sigma}^{\delta}$ and $\bar{\pi}^{\delta}$. This approximation procedure is the fourth we are considering here.

The quantization scheme presented above depends largely on k. As in the next section, we only consider L^2-approximations of x, it is then more convenient to construct $\bar{x}$ and the corresponding filters for k = 2.

6. EFFECT OF THE APPROXIMATIONS OF THE SIGNAL PROCESS AND OF THE QUANTIZATION OF THE OBSERVATION PROCESS

Since we only need L^2-norms for the evaluation of a bound of $||Z_t - \hat{Z}_t||_{0,1}$, all the constants $K_1(k)$, $K_2(k)$, $\hat{K}(k)$, $\bar{K}(k)$ etc. of the preceding section will simply be written without the argument k as $K_1, K_2, \hat{K}, \bar{K}$ etc. Similarly, $||\cdot||_2$ will be denoted by $||\cdot||$. The same symbol is also used for $||F||$ as the sup-norm of F. We shall avoid any possible ambiguity by clarifying the meaning when we shall deal with the sup-norm.

We go back to Formula (4.10) where the superscript $^\circ$ represents the modified objects of the first approximation, obtained after x is replaced by $\hat{x}$ or $\bar{x}$, constructed in Section 5, with k=2.

We need to evaluate a bound for (4.20) and we first deal with $||C_s - \mathcal{C}_{t,s}||$. We have, for $(n-1)\delta < s \leq n\delta \leq t$,

$$(6.1) \qquad ||C_s - \mathcal{C}_{t,s}|| = ||h(s,x_s) - \frac{1}{\delta}\int_{(n-1)\delta}^{n\delta} h(u,\overset{\circ}{x}_{(n-1)\delta})du|| \leq$$

$$\leq [\frac{1}{\delta}\int_{(n-1)\delta}^{n\delta} E|h(s,x_s) - h(u,\overset{\circ}{x}_{(n-1)\delta})|^2 du]^{\frac{1}{2}} \leq$$

$$\leq K_0 \{\delta(1 + ||x_s|| + ||\overset{\circ}{x}_{(n-1)\delta}||) + ||x_s - x_{(n-1)\delta}|| +$$

$$+ ||x_{(n-1)\delta} - \hat{x}_{(n-1)\delta}|| + \epsilon||\hat{x}_{(n-1)\delta} - \bar{x}_{(n-1)\delta}||\}$$

where $\epsilon = 0$ if $\overset{\circ}{x} = \hat{x}$ and $\epsilon = 1$ if $\overset{\circ}{x} = \bar{x}$.

As $||\bar{x}_{n\delta}|| \leq ||\hat{x}_{n\delta}|| + ||\hat{x}_{n\delta} - \bar{x}_{n\delta}||$, according to (5.15), we can write

$$(6.2) \qquad ||\bar{x}_{n\delta}|| \leq K_1 + \bar{K}\sqrt{\delta}$$

Therefore (6.1) becomes

$$||C_s - \mathcal{C}_{t,s}|| \leq K_0\{\delta(1+2K_1 + \epsilon\bar{K}\sqrt{\delta}) + (K_2 + \hat{K} + \epsilon\bar{K})\sqrt{\delta}\}$$

where $\hat{K}$ is given by (5.9).

Consequently,

$$(6.3) \qquad \| C - \mathcal{C}_t \|_t \leq \{A_1 + K_0\sqrt{T}\,[\epsilon\bar{K}\,(1+\delta) + \hat{K}]\}\,\sqrt{\delta}$$

where A_1 is given by (4.25).

Similarly, we have

$$(6.4)\qquad ||C^{\delta}_{t,s} - \mathring{C}_{t,s}|| = ||\frac{1}{\delta}\int_{(n-1)\delta}^{n\delta}[h(u,x_u) - h(u,\mathring{x}_{(u-1)\delta})]du|| \le$$

$$\le K_0 \sup_u ||x_u - \mathring{x}_{(n-1)\delta}|| \le K_0(K_2 + \hat{K} + \epsilon\bar{K})\sqrt{\delta}$$

where $(n-1)\delta < u \le n\delta \le t$. Therefore

$$(6.5)\qquad \| C^{\delta}_t - \mathring{C}_t \|_t \le \sqrt{T}K_0(K_2 + \hat{K} + \epsilon\bar{K})\sqrt{\delta}$$

We also have

$$(6.6)\qquad \| C \|_t + \| \mathring{C}_t \|_t \le \sqrt{T}[K_0(1+K_1) + K_0(1+K_1 + \epsilon\bar{K}\sqrt{\delta})] =$$

$$= 2\sqrt{T}\, K_0(1+K_1) + \epsilon\sqrt{T}\, K_0\, \bar{K}\sqrt{\delta}$$

and

$$(6.7)\qquad \| \mathring{C}_t \|_t \le \sqrt{T}K_0(1+K_1 + \epsilon\bar{K}\sqrt{\delta})$$

By bringing (6.3 ----7) and (4.25) into (4.20) we get

$$(6.8)\qquad ||Z_t - \mathring{Z}_t||_1\, \delta^{-\frac{1}{2}}$$

$$\le A_1 + \{A_1 + \epsilon' K_0 \sqrt{T}\, [\epsilon\bar{K}(1+\delta) + \hat{K}]\}\ \{1 + \tfrac{1}{2}\sqrt{\delta}\{A_1 + \epsilon' K_0\sqrt{T}[\epsilon\bar{K}(1+\delta) + \hat{K}]\} +$$

$$+ \sqrt{T}\, K_0\, (1+K_1) + \tfrac{1}{2}\epsilon\sqrt{T}\, K_0\bar{K}\sqrt{\delta}\}$$

$$+ \epsilon'\sqrt{T}\, K_0(K_2 + \hat{K} + \epsilon\bar{K})\sqrt{\delta}\ [1+\sqrt{T}K_0(1+K_1+\epsilon\bar{K}\sqrt{\delta})]$$

$$=: A^{\delta}_1\ (\epsilon,\epsilon')$$

where $\epsilon' = 1$.

For the evaluation of a bound of (4.10) we need to compute a bound for $||F(x_t) - F(\mathring{x}_t)||_1$. We then suppose that F is uniformly Lipschitz continuous i.e.,

$$(6.9)\qquad |F(x) - F(x')| \le K|x-x'|, \qquad \forall x,x' \in \mathbb{R}^q$$

with a constant K.

We therefore have

$$(6.10)\qquad ||F(x_t)-F(\mathring{x}_t)||_1 \le K||x_t-\mathring{x}_t||_2 \le \epsilon' K(K_2+\hat{K}+\epsilon\bar{K})\sqrt{\delta} =: A^{\delta}_2(\epsilon',\epsilon)\sqrt{\delta}$$

where $\epsilon' = 1$.

Finally we have the following extension of Theorem 4.3.

THEOREM 6.1

Under hypotheses (2.4), (4.21), (6.9) and $x_0 \in L^2$, we have

$$(6.11) \qquad \|\pi(F) - \mathring{\pi}(F)\|_1 \leq [2\|F\| A_1^\delta(\varepsilon',\varepsilon) + A_2^\delta(\varepsilon',\varepsilon)]\sqrt{\delta}$$

where $\|F\|$ is the sup-norm of F, A_1^δ and A_2^δ are given by (6.8) and (6.10), respectively.

For $\varepsilon'=0$, the right hand side of the inequality gives a bound of $\|\pi(F)-\pi^\delta(F)\|_1$, (but in this case the Lipschitz continuity of F is not needed), for $\varepsilon'=1$ and $\varepsilon=0$ it gives a bound for $\|\pi(F)-\hat{\pi}^\delta\|_1$ and for $\varepsilon'=1$ and $\varepsilon=1$ a bound for $\|\pi(F)-\bar{\pi}^\delta\|_1$. □

We recall that the quantization of $\hat{x}$ was made under the supplementary hypothesis that the distribution function of x_0 is continuous.

If the values taken by $(y_{n\delta})$ were exactly known, then the approximate filter $\bar{\pi}^\delta$ could be computed with a desired accuracy by a finite number of operations. But the observation is usually measured in terms of units of a finite set; more precisely, measurement devices offer only quantized values of the observation. The problem is to characterize a quantization of (y_n^δ) and therefore of $(y_{n\delta})$ which would not affect the degree of approximation of π.

Let us put

$$(6.12) \qquad \bar{Z}_t = \exp \sum_{n=0}^{[t/\delta]} (\bar{H}_n^\delta y_n^\delta - \frac{\delta}{2} |\bar{H}_n^\delta|^2)$$

where $\bar{H}_n^\delta$ is given by

$$(6.13) \qquad \bar{H}_n^\delta = \frac{1}{\delta} \int_{(n-1)\delta}^{n\delta} h(s,\bar{x}_{(n-1)\delta})ds$$

and $\bar{x}$ is the quantized approximation of x for k=2.

$\bar{Z}$ is the likelihood ratio corresponding to the fourth approximation procedure. We shall replace y_n^δ by a finite space valued random variable approximating it. More precisely, we choose the real step function

(6.14) $$C(x) = \sum_{i=1}^{J-1} c_j 1_{]v_j, v_{j+1}]}(x), \quad v_0 < v_1 < \ldots < v_J$$

with finite J and define $\bar{y}_n^\delta = (\bar{y}_n^{\delta 1}, \ldots, y_n^{\delta p})$ by

(6.15) $$\bar{y}_n^{\delta, i} = C(y_n^{\delta, i})$$

in such a way that if y_n^δ is replaced by $\bar{y}_n^\delta$ in (6.12) the corresponding filter $\bar{\bar{\pi}}^\delta$ should be close to π.

We put

(6.16) $$||y_n^\delta - \bar{y}_n^\delta||_{0,4} = \varphi\delta^{\frac{3}{2}} \quad \text{and} \quad c = \sup_j c_j$$

(6.17) $$\bar{\bar{Z}}_t = \exp \sum_{n=0}^{[t/\delta]} (\bar{H}_n^\delta \bar{y}_n^\delta - \frac{\delta}{2} | \bar{H}_n^\delta|^2)$$

and define $\bar{\bar{\sigma}}_t^\delta$ and $\bar{\bar{\pi}}_t^\delta$ by

(6.18) $$\bar{\bar{\sigma}}_t^\delta(F) = E^Q[\bar{\bar{Z}}_t F(\bar{x}_t)] \quad \text{and} \quad \bar{\bar{\pi}}_t^\delta(F) = \bar{\bar{\sigma}}_t^\delta(F)/\bar{\bar{\sigma}}_t^\delta(1).$$

for all $F \in \beta(\mathbb{R}^q)$.

We remark that $\bar{\bar{Z}}_t$ is not the likelihood ratio corresponding to $\bar{y}^\delta$, but $\bar{\bar{\pi}}_t^\delta$ is still a probability distribution on the finite space of values of $\bar{x}$.

In order to simplify the problem we first suppose that h is bounded.

Formulas (4.7) and (4.8) are still valid with $\overset{\circ}{\sigma}$ and $\overset{\circ}{\pi}$ replaced by $\bar{\bar{\sigma}}^\delta$ and $\bar{\bar{\pi}}^\delta$, and we can write

(6.19) $$||\pi(F) - \bar{\bar{\pi}}_t^\delta(F)||_1 \le 2||F|| \, ||Z_t - \bar{\bar{Z}}_t||_{0,1} + ||\bar{\bar{Z}}_t[F(x_t) - F(\bar{x}_t)]||_{0,1}.$$

Putting

$$||Z_t - \bar{\bar{Z}}_t||_{0,1} \le ||Z_t - \bar{Z}_t||_{0,1} + || \bar{Z}_t - \bar{\bar{Z}}_t||_{0,1}$$
$$\bar{\bar{Z}}_t = \bar{\bar{Z}}_t - \bar{Z}_t + \bar{Z}_t$$

and bringing into (6.19) we can write

(6.20) $$||\pi_t(F) - \bar{\bar{\pi}}_t^\delta(F)||_1 \le 2||F|| \, ||Z_t - \bar{Z}_t||_{0,1} + ||F(x_t) - F(\bar{x}_t)||_1$$
$$+ 2||F|| \, ||\bar{Z}_t - \bar{\bar{Z}}_t||_{0,1} + ||(\bar{Z}_t - \bar{\bar{Z}}_t) \, [F(x_t) - F(\bar{x}_t)]||_{0,1}$$

Notice that the first two terms of the right hand side dominate

$||\pi_t(F)-\bar{\pi}_t^{\delta}(F)||_1$ and its bound is given in Theorem 6.1. Therefore, we only need to compute a reasonable bound for the sum of the third and fourth terms. According to (6.10), this sum is bounded by

$$(6.21)\qquad ||\bar{Z}_t-\bar{\bar{Z}}_t||_{0,2}(2||F|| + K||x_t-\bar{x}_t||_2)$$
$$\leq ||\bar{Z}_t - \bar{\bar{Z}}_t||_{0,2}\ [2||F|| + K(K_2+\hat{K}+\bar{K})\sqrt{\delta}]$$

Therefore we need to make $||\bar{Z}_t-\bar{\bar{Z}}_t||_{0,2}$ small.

By putting $||h|| = \sup_{t,x} |h(t,x)|$ we can write

$$||\bar{Z}_t-\bar{\bar{Z}}_t||_{0,2} \leq ||\sum_{n=0}^{[t/\delta]} \bar{H}_n^{\delta}(y_n^{\delta} - \bar{y}_n^{\delta})||_{0,4}\ ||\bar{Z}_t+\bar{\bar{Z}}_t||_{0,4}$$
$$\leq T||h||\varphi\sqrt{\delta}\ (||\bar{Z}_t||_{0,4} + ||\bar{\bar{Z}}_t||_{0,4})$$

We have

$$||\bar{Z}_t||_{0,4} \leq \exp(\tfrac{3}{2}T\ ||h||^2) =: K_3 \quad \text{and}$$
$$\bar{\bar{Z}}_t^4 = \exp[4\sum_{n=0}^{[t/\delta]}(\bar{H}_n^{\delta}\ \bar{y}_n^{\delta} - \frac{\delta}{2}\ |\bar{H}_n^{\delta}|^2)] \leq \exp\frac{4cT||h||}{\delta}$$

Therefore

$$(6.22)\qquad ||\bar{Z}_t-\bar{\bar{Z}}_t||_{0,2} \leq T||h||\varphi\sqrt{\delta}\ [K_3 + \exp\frac{cT||h||}{\delta}]$$

The quantization can be carried out in such a way that

$$(6.23)\qquad \varphi = D[K_3 + \exp\frac{cT||h||}{\delta}]^{-1}$$

where D is a constant. We then obtain a bound for $||\pi(F) - \bar{\bar{\pi}}^{\delta}(F)||_1$, proportional to $\sqrt{\delta}$.

Therefore we can state the following

THEOREM 6.2

Under the hypotheses of Theorem 6.1 and for bounded h, variables y_n^{δ}; $n=1,2,\ldots,[T/\delta]$, can be approximated by finite space valued $\bar{y}_n^{\delta}$, as defined by (6.15), in such a way that (6.23) is satisfied. In this case if y_n^{δ} is replaced by $\bar{y}_n^{\delta}$ in the expression of $\bar{\pi}^{\delta}$ yielding a new filter $\bar{\bar{\pi}}^{\delta}$, then

$$\|\pi(F)-\bar{\bar{\pi}}^{\delta}(F)\|_1 \le 2\|F\|[A_1^{\delta}(1,1) + A_2^{\delta}(1,1)]\sqrt{\delta} + \tag{6.24}$$
$$+ TD\|h\|[2\|F\| + K(K_2+\hat{K} + \bar{K})\sqrt{\delta}]\sqrt{\delta}$$

where the first term of the right hand side is the bound of $\|\pi(F) - \bar{\pi}^{\delta}(F)\|_1$ given by (6.11) for $\epsilon'=\epsilon=1$ and D is a conveniently chosen constant for the fulfillment of (6.23). □

The procedure leading to $\bar{\bar{\pi}}^{\delta}$ is the fifth and the last of the approximation procedures considered here.

REMARK 6.3

In the case where h is bounded, the second inequality of (4.21) is not necessary and corresponding modifications of A_1^{δ} and A_2^{δ} can be easily made. But it is always possible to get rid of the boundedness condition of h in the last theorem. In fact $\bar{\pi}^{\delta}$ can be approximated by a truncation of h in such a way that if $\bar{\pi}^{tr}$ is the filter of the fourth approximation corresponding to the truncated h say h^{tr}, then $\|\bar{\pi}(F) - \bar{\pi}^{tr}(F)\|_1$ is bounded by a quantity proportional to $\sqrt{\delta}$. In this case the fifth approximate filter can be constructed with h^{tr} and give again for $\|\pi(F) - \bar{\bar{\pi}}^{\delta}(F)\|_1$ a bound proportional to $\sqrt{\delta}$.

REFERENCES

[1] P. Brémaud and M. Yor (1978), Changes of filtration and of probability measures, Z. Wahrschein. verw. Geb. 45, 269-295.

[2] C. Dellacherie and P.-A. Meyer (1982), Probabilities and Potential B, North Holland.

[3] G.B. Di Masi, M. Pratelli and W.J. Runggaldier (1985), An approximation for the nonlinear filtering problem with error bound, Stochastics 14, 247-271.

[4] A. Friedman (1975), Stochastic Differential Equations and Applications, Vol. 1, Academic Press.

[5] M. Fujisaki, G. Kallianpur and H. Kunita (1972), Stochastic differential equations for the nonlinear filtering problem, Osaka J. Math. 9, 14-40.

[6] I.I. Gihman and A.V. Skorohod (1972), Stochastic Differential Equations, Springer-Verlag.

[7] M. Hazewinkel and J.C. Willems (eds.) (1981), Stochastic Systems: The Mathematics of Filtering Identification with Applications, D. Reidel Publishing Co., Dordrecht.

[8] A.H. Jazwinski (1970), Stochastic Processes and Filtering, Academic Press.

[9] G. Kallianpur (1980), Stochastic Filtering Theory, Springer-Verlag.

[10] G. Kallianpur and R.L. Karandikar (1985), White noise calculus and nonlinear filtering theory, The Annals of Probability, 13(4), 1033-1107.

[11] H. Korezlioglu and G. Mazziotto (1984), Approximations of the nonlinear filter by periodic sampling and quantization, Analysis and Optimization of Systems, Part 1, Lect. Notes in Cont. and Inf. Sc. 62, Springer-Verlag.

[12] H. Korezlioglu, G. Mazziotto and J. Szpirglas (eds.) (1984), Filtering and Control of Random Processes, Lect. Notes in Cont. and Inf. Sc. 61, Springer-Verlag.

[13] H.J. Kushner (1977), Probability Methods for Approximations in Stochastic Control and for Elliptic Equations, Academic Press.

[14] R.S. Liptser and A.N. Shiryaev (1977, 1978), Statistics of Random Processes, Vol. 1 General Theory, Vol. 2: Applications, Springer-Verlag.

[15] G.N. Mil'shtein (1974), Approximate integration of stochastic differential equations, Theory Prob. Appl. XIX-3, 557-562.

[16] S.K. Mitter and A. Moro (eds.) (1983). Nonlinear Filtering and Stochastic Control, Lect. Notes in Math., 972, Springer-Verlag.

[17] J. Neveu (1972), Martingales à Temps Discret, Masson, Paris.

[18] E. Pardoux and D. Talay (1985), Discretization and simulation of stochastic differential equations, Acta Applicandae Mathematicae 3, 23-47.

[19] J. Picard (1984), Approximation of nonlinear filtering problems and order of convergence, Filtering and Control of Random Processes, Lect. Notes in Cont. and Inf. Sc. 61, Springer-Verlag.

[20] J. Picard (1986), An estimate of the error in time discretization of nonlinear filtering problems, Theory and Applications of Nonlinear Control Systems, North-Holland.

[21] E. Platen (1981), Approximation method for a class of Itô processes, Lietuvos Mat. Rinkinys XXI-1, 121-133.

[22] J. Szpirglas and G. Mazziotto (1979), Modèle général de filtrage non-Linéaire et equations differentielles stochastiques associees, Ann. Inst. Henri Poincare, B 15(2), 147-173.

[23] J.H. Van Schuppen, Stochastic Filtering Theory: A discussion of concepts, methods and results, Stochastic Control Theory and Stochastic Differential Systems, Lect. Notes in Cont. and Inf. Sc. 16, 209-226, Springer-Verlag.

[24] Y. Yavin (1985), Numerical Studies in Nonlinear Filtering, Lect. Notes in Cont. and Inf. Sc. 65, Springer-Verlag.

[25] M. Zakai (1969), On the optimal filtering of diffusion processes, Z. Wahrschein. verw. Geb. 11, 230-249.

END
DATE
FILMED
6-1988
DTIC